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Lie Symmetries of Quadratic Two-Dimensional Difference Equations

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We find all systems of first-order quadratic autonomous two-dimensional difference equations which have two linear Lie symmetries. Knowledge of these symmetries permits the systems to be integrated by a reduction procedure.

The identification of integrable systems for continuous or discrete equations is an important problem in applied mathematics. Discrete dynamical systems have been studied in many contexts in the recent years. They appear in discretization procedures of continuous systems, or, more naturally, in models described within a discrete space, for instance, in many biological systems. Two-dimensional continuous systems of first-order autonomous ordinary differential equations have no chaotic behavior; however, there are chaotic two-dimensional autonomous difference equations, Hénon's map, for example (Hénon, 1976). In many cases, discretization of completely integrable continuous systems also can exhibit chaotic behavior (Date *et al.*, 1982).

Although integrable discrete systems have been known for decades (MacMillan, 1971), few systematic studies were undertaken in this direction (Hirota, 1979; Maeda, 1987; Grammaticos *et al.*, 1991; Quispel and Sahadevan, 1993). We study here two-dimensional systems of first-order quadratic autonomous difference equations. These equations are discrete counterparts of Lotka–Volterra continuous systems, which are important in population dynamics (Gardini *et al.*, 1987). We analyze the invariance of these discrete equations under a continuous group of symmetries for determining integrable cases. Maeda (1987) extended Lie's algorithm for finding symmetries of difference equations and constructed a procedure for making a reduction of

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an autonomous system if it has a Lie symmetry. Quispel and Sahadevan (1993) extended this method to nonautonomous systems.

We summarize the method formulated by Quispel and Sahadevan (1993). Consider a system of coupled difference equations

$$x^i_{n+1} = \Phi^i(x^j_n, n) \tag{1}$$

where Φ^i are given functions and $i = 1, \ldots, N$. The infinitesimal transformation

$$x'^i_n = x^i_n + \epsilon\xi^i(x^j, n) \tag{2}$$

is a symmetry transformation if

$$\xi^i(\Phi(x^j_n, n), n + 1) = \sum_{k=1}^{N} \xi^k(x^j_n, n) \frac{\partial}{\partial x_k} (\Phi^i(x^j_n, n)) \tag{3}$$

The symmetry operator is

$$\mathbf{U} = \sum_{j=1}^{N} \xi^j(x, n) \frac{\partial}{\partial x^j} \tag{4}$$

We now apply a coordinate transformation $y = y(x)$ in the system (1), which brings the system to the form

$$y^i_{n+1} = \Psi^i(y^j_n, n) \tag{5}$$

and the symmetry operator is changed to

$$\mathbf{U} = \sum_{i=1}^{N} \xi'^j \frac{\partial}{\partial y^j} \tag{6}$$

where

$$\xi'^i(y, n) = \xi^j(x, n) \frac{\partial}{\partial x^j} (y^i) \tag{7}$$

If a Lie symmetry exists, we can choose a coordinate system where

$$\xi'^j(y, n) = \delta^j_1 \tag{8}$$

and where the symmetry operator takes the form

$$\mathbf{U} = \frac{\partial}{\partial y^1} \tag{9}$$

In this case the conditions for symmetry (3) are written as

$$\frac{\partial}{\partial y^1} (\Psi^i) = \delta^i_1 \tag{10}$$

and after integration we have

$$\Psi^i(y, n) = y^1 + \Theta^i(y^2, \ldots, y^N, n), \qquad i = 1, \ldots, N - 1$$
$$\Psi^N(y, n) = \Theta^N(y^2, \ldots, y^N, n) \tag{11}$$

We see that in the new y coordinates the $N - 1$ variables $y^2, \ldots, y^N$ form a new system, and equation (1) is decoupled and can be integrated. If we have a new symmetry, we need to solve the system

$$\mathbf{U}y^j = \delta^j_1 \tag{12}$$

to find the expected coordinates. If $N = 2$, we need two linear independent Lie symmetries for integrating the system.

We will consider the system

$$x_{n+1} = a_{00} + a_{10}x_n + a_{01}y_n + a_{11}x_ny_n + a_{20}x_n^2 + a_{02}y_n^2$$
$$y_{n+1} = b_{00} + b_{10}x_n + b_{01}y_n + b_{11}x_ny_n + b_{20}x_n^2 + b_{02}y_n^2 \tag{13}$$

We make the ansatz that the symmetries have linear dependence in x_n and y_n:

$$\xi^1 = c_0(n) + c_1(n)x_n + c_2(n)y_n$$
$$\xi^2 = d_0(n) + d_1(n)x_n + d_2(n)y_n \tag{14}$$

By applying equations (7), we get the following system of linear equations to be solved:

$$0 = c_0(n+1) + c_1(n+1)a_{00} + c_2(n+1)b_{00} - a_{10}c_0(n) - a_{01}d_0(n)$$
$$0 = -a_{11}c_0(n) + c_1(n+1)a_{01} + c_2(n+1)b_{01} - a_{01}d_2(n)$$
$$- 2a_{02}d_0(n) - a_{10}c_2(n)$$
$$0 = c_2(n+1)b_{02} + c_1(n+1)a_{02} - a_{11}c_2(n) - 2a_{02}d_2(n)$$
$$0 = c_1(n+1)a_{10} - a_{10}c_1(n) - 2a_{20}c_0(n) - a_{11}d_0(n)$$
$$- a_{01}d_1(n) + c_2(n+1)b_{10}$$
$$0 = c_1(n+1)a_{11} - 2a_{20}c_2(n) - a_{11}c_1(n) + c_2(n+1)b_{11}$$
$$- 2a_{02}d_1(n) - a_{11}d_2(n)$$
$$0 = c_1(n+1)a_{20} - 2a_{20}c_1(n) + c_2(n+1)b_{20} - a_{11}d_1(n) \tag{15}$$
$$0 = d_0(n+1) + d_1(n+1)a_{00} + d_2(n+1)b_{00} - b_{10}c_0(n) - b_{01}d_0(n)$$
$$0 = -b_{11}c_0(n) + d_1(n+1)a_{01} + d_2(n+1)b_{01} - b_{01}d_2(n)$$
$$- 2b_{02}d_0(n) - b_{10}c_2(n)$$
$$0 = d_2(n+1)b_{02} + d_1(n+1)a_{02} - b_{11}c_2(n) - 2b_{02}d_2(n)$$

$$0 = d_1(n+1)a_{10} - b_{10}c_1(n) - 2b_{20}c_0(n) - b_{11}d_0(n)$$
$$- b_{01}d_1(n) + d_2(n+1)b_{10}$$
$$0 = d_1(n+1)a_{11} - 2b_{20}c_2(n) - b_{11}c_1(n) + d_2(n+1)b_{11}$$
$$- 2b_{02}d_1(n) - b_{11}d_2(n)$$
$$0 = d_1(n+1)a_{20} - 2b_{20}c_1(n) + d_2(n+1)b_{20} - b_{11}d_1(n)$$

With the help of algebraic computation we solved this system and found several cases with two linear independent Lie symmetries (Table I). Some of these cases can be integrated trivially. As an example of application of the reduction method we will solve here case 10. The solution of other cases which are not trivially integrated are given in Table II.

We apply conditions (8) in $\mathbf{U}_1$ to obtain the first reduction in case 10:

$$1 = 2^n\left(x_n \frac{\partial}{\partial y_n}(X_n) + y_n \frac{\partial}{\partial y_n}(X_n)\right)$$
$$0 = 2^n\left(x_n \frac{\partial}{\partial y_n}(Y_n) + y_n \frac{\partial}{\partial y_n}(Y_n)\right) \tag{16}$$

where X_n and Y_n are the sought new (first) coordinates. These equations are easily solved to give

$$X_n = \frac{\ln(2^n x_n)}{2^n}$$
$$Y_n = \frac{y_n}{x_n} \tag{17}$$

The transformed system is

$$X_{n+1} = X_n + 2^{-n-1}\ln\left(\frac{2Y_n(-2 + a_{02}Y_n)}{2^n}\right)$$
$$Y_{n+1} = \frac{1 + Y_n^2}{Y_n(-2 + a_{02}Y_n)} \tag{18}$$

We applied conditions (8) in $\mathbf{U}_2$ to obtain the second reduction:

$$0 = \frac{(-1)^n}{2}(2Y_n + a_{02})\frac{\partial}{\partial X_n}(V_n) - (-2)^n(-1 + a_{02}Y_n + Y_n^2)\frac{\partial}{\partial Y_n}(V_n)$$
$$1 = \frac{(-1)^n}{2}(2Y_n + a_{02})\frac{\partial}{\partial X_n}(W_n) - (-2)^n(-1 + a_{02}Y_n + Y_n^2)\frac{\partial}{\partial Y_n}(W_n) \tag{19}$$

where V_n and W_n are the second transformed coordinates.

Table I.

No.	Mappings	Symmetry vector fields
1	$x_{n+1} = a_{00} + x_n + a_{01} y_n + y_n^2$ $y_{n+1} = \frac{a_{01}(-1 + b_{01})}{2} + b_{01} y_n$	$\mathbf{U}_1 = \frac{\partial}{\partial x_n}$ $\mathbf{U}_2 = \left(-\frac{(4a_{00} - a_{01}^2)(n+1)}{4} + x_n\right)\frac{\partial}{\partial x_n} + \left(\frac{a_{01}}{4} + \frac{y}{2}\right)\frac{\partial}{\partial y_n}$
2	$x_{n+1} = \frac{a_{10}(a_{10} - 2)}{4} + a_{10} x_n + x_n^2$ $y_{n+1} = b_{00} + b_{01} y_n$	$\mathbf{U}_1 = 2^n\left(\frac{a_{10}}{2} + x_n\right)\frac{\partial}{\partial x_n} + \left(\frac{b_{00}}{-1 + b_{01}} + y_n\right)\frac{\partial}{\partial y_n}$ $\mathbf{U}_2 = b_{01}^n \frac{\partial}{\partial y_n}$
3	$x_{n+1} = \frac{a_{10}(a_{10} - 2)}{4} + a_{10} x_n + x_n^2$ $y_{n+1} = b_{00} + y_n$	$\mathbf{U}_1 = 2^n\left(\frac{a_{10}}{2} + x_n\right)\frac{\partial}{\partial x_n} + (1 - b_{00} n + y_n)\frac{\partial}{\partial y_n}$ $\mathbf{U}_2 = \frac{\partial}{\partial y_n}$
4	$x_{n+1} = a_{10} x_n$ $y_{n+1} = x_n y_n$	$\mathbf{U}_1 = x_n \frac{\partial}{\partial x_n} + (n - 1) y_n \frac{\partial}{\partial y_n}$ $\mathbf{U}_2 = a_{10}^n \frac{\partial}{\partial x_n}$
5	$x_{n+1} = x_n^2$ $y_{n+1} = 2x_n y_n + y_n^2$	$\mathbf{U}_1 = 2^n x_n\left(\frac{\partial}{\partial x_n} - \frac{\partial}{\partial y_n}\right)$ $\mathbf{U}_2 = 2^n(x_n + y_n)\frac{\partial}{\partial y_n}$
6	$x_{n+1} = y_n^2$ $y_{n+1} = x_n^2$	$\mathbf{U}_1 = 2^n\left(x_n \frac{\partial}{\partial x_n} + y_n \frac{\partial}{\partial y_n}\right)$ $\mathbf{U}_2 = (-2)^{n-1}\left(x_n \frac{\partial}{\partial x_n} - y_n \frac{\partial}{\partial y_n}\right)$
7	$x_{n+1} = x_n^2$ $y_{n+1} = b_{11} x_n y_n$	$\mathbf{U}_1 = 2^n\left(x_n \frac{\partial}{\partial x_n} + \frac{\partial}{\partial y_n}\right)$ $\mathbf{U}_2 = \frac{(-1)^n}{3}\left(-x_n \frac{\partial}{\partial x_n} + 2y_n \frac{\partial}{\partial y_n}\right)$
8	$x_{n+1} = x_n y_n$ $y_{n+1} = x_n y_n - y_n^2$	$\mathbf{U}_1 = 2^n\left(x_n \frac{\partial}{\partial x_n} + y_n \frac{\partial}{\partial y_n}\right)$ $\mathbf{U}_2 = -\frac{(-1)^n}{3}\left[(2x_n - 3y_n)\frac{\partial}{\partial x_n} - y_n \frac{\partial}{\partial y_n}\right]$

Table I. Continued.

No.	Mappings	Symmetry vector fields
9	$x_{n+1} = x_n y_n$ $y_{n+1} = b_{20} x_n^2$	$\mathbf{U}_1 = 2^n\left(x_n \frac{\partial}{\partial x_n} + y_n \frac{\partial}{\partial y_n}\right)$ $\mathbf{U}_2 = \frac{(-1)^n}{3}\left(-x_n \frac{\partial}{\partial x_n} + 2y_n \frac{\partial}{\partial y_n}\right)$
10	$x_{n+1} = -2x_n y_n + a_{02} y_n^2$ $y_{n+1} = x_n^2 + y_n^2$	$\mathbf{U}_1 = 2^n\left(x_n \frac{\partial}{\partial x_n} + y_n \frac{\partial}{\partial y_n}\right)$ $\mathbf{U}_2 = (-2)^{n-1}\left[(a_{02} x_n + 2y_n) \frac{\partial}{\partial x_n} - (a_{02} y_n - 2x_n) \frac{\partial}{\partial y_n}\right]$
11	$x_{n+1} = 2x_n y_n$ $y_{n+1} = x_n^2 + y_n^2$	$\mathbf{U}_1 = 2^n\left(x_n \frac{\partial}{\partial x_n} + y_n \frac{\partial}{\partial y_n}\right)$ $\mathbf{U}_2 = -2^n\left(y_n \frac{\partial}{\partial x_n} + x_n \frac{\partial}{\partial y_n}\right)$
12	$x_{n+1} = x_n^2$ $y_{n+1} = y_n^2$	$\mathbf{U}_1 = 2^n\left(x_n \frac{\partial}{\partial x_n} + 2y_n \frac{\partial}{\partial y_n}\right)$ $\mathbf{U}_2 = -(-2)^{n-1}\left[(a_{02} x_n + 2y_n) \frac{\partial}{\partial x_n} - (a_{02} y_n - 2x_n) \frac{\partial}{\partial y_n}\right]$
13	$x_{n+1} = -2x_n y_n - 4y_n^2$ $y_{n+1} = x_n y_n + \frac{x_n^2}{8} + y_n^2$	$\mathbf{U}_1 = 2^n\left(x_n \frac{\partial}{\partial x_n} + y_n \frac{\partial}{\partial y_n}\right)$ $\mathbf{U}_2 = -2^n\left(y_n \frac{\partial}{\partial x_n} - \frac{(x_n + 4y_n)}{8} \frac{\partial}{\partial y_n}\right)$

The solution of these equations is

$$V_n = e^{-X_n}(-1 + a_{02} Y_n + Y_n^2)^{-2^{-n-1}}$$

$$W_n = \frac{(-2)^{-n}}{\sqrt{4 + a_{02}^2}} \ln\left(-\frac{2Y_n + a_{02} + \sqrt{4 + a_{02}^2}}{-\sqrt{4 + a_{02}^2} + 2Y_n + a_{02}}\right) \tag{20}$$

The second transformed difference system is linear

$$V_{n+1} = (-1)^{(-2)^{-n} 2^{(-1+n)/2^{n+1}}} V_n$$

$$W_{n+1} = W_n + \frac{(-2)^{-n-1}}{\sqrt{4 + a_{02}^2}} \ln \frac{a_{02}^2 + \sqrt{4 + a_{02}^2}\, a_{02} + 2}{-a_{02}^2 + \sqrt{4 + a_{02}^2}\, a_{02} - 2} \tag{21}$$

Table II.

No.	Integrated systems
8	$\begin{aligned} x_n &= -(-y_0)^{[(-1)^n+2^{n+1}]/3}(x_0 - y_0)^{[(-1)^{n+1}+2^n]/3} \\ &\quad + (-y_0)^{[2(-1)^{n+1}+2^{n+1}]/3}(x_0 - y_0)^{[2(-1)^n+2^n]/3} \\ y_n &= -(-y_0)^{[(-1)^n+2^{n+1}]/3}(x_0 - y_0)^{[(-1)^{n+1}+2^n]/3} \end{aligned}$
10	$\begin{aligned} x_n &= -\frac{i(x_0 a_{02} y_0 - x_0^2 + y_0^2)^{2^{-1+n}}}{\sqrt{4 + a_{02}^2}} \\ &\quad \times \Bigg[\Bigg(-\frac{2(4 + a_{02}^2 + \sqrt{4 + a_{02}^2}\,a_{02})x_0 + 4\sqrt{4 + a_{02}^2}\,y_0}{(4 + a_{02}^2 + \sqrt{4 + a_{02}^2}\,a_{02})(-a_{02}^2 + \sqrt{4 + a_{02}^2}\,a_{02} - 2)x_0 + 4\sqrt{4 + a_{02}^2}\,y_0}\Bigg)^{-(-2)^{-1+n}} \\ &\quad \times \Bigg(\frac{-a_{02}^2 + \sqrt{4 + a_{02}^2}\,a_{02} - 2}{a_{02}^2 + \sqrt{4 + a_{02}^2}\,a_{02} + 2}\Bigg)^{[(-2)^n-1]/6} \\ &\quad + \Bigg(-\frac{2(4 + a_{02}^2 + \sqrt{4 + a_{02}^2}\,a_{02})x_0 + 4\sqrt{4 + a_{02}^2}\,y_0}{(4 + a_{02}^2 + \sqrt{4 + a_{02}^2}\,a_{02})(-a_{02}^2 + \sqrt{4 + a_{02}^2}\,a_{02} - 2)x_0 + 4\sqrt{4 + a_{02}^2}\,y_0}\Bigg)^{-(-2)^{-1+n}} \\ &\quad \times \Bigg(\frac{a_{02}^2 + \sqrt{4 + a_{02}^2}\,a_{02} + 2}{-a_{02}^2 + \sqrt{4 + a_{02}^2}\,a_{02} - 2}\Bigg)^{[(-2)^n-1]/6}\Bigg] \\ y_n &= \frac{i(x_0 a_{02} y_0 - x_0^2 + y_0^2)^{2^{-1+n}}(4 + a_{02}^2 + \sqrt{4 + a_{02}^2}\,a_{02})}{16 + 4a_{02}^2} \\ &\quad \times \Bigg(\Bigg(-\frac{2(4 + a_{02}^2 + \sqrt{4 + a_{02}^2}\,a_{02})x_0 + 4\sqrt{4 + a_{02}^2}\,y_0}{(4 + a_{02}^2 + \sqrt{4 + a_{02}^2}\,a_{02})(-a_{02}^2 + \sqrt{4 + a_{02}^2}\,a_{02} - 2)x_0 + 4\sqrt{4 + a_{02}^2}\,y_0}\Bigg)^{-(-2)^{-1+n}} \\ &\quad \times \Bigg(\frac{-a_{02}^2 + \sqrt{4 + a_{02}^2}\,a_{02} - 2}{a_{02}^2 + \sqrt{4 + a_{02}^2}\,a_{02} + 2}\Bigg)^{[(-2)^n-1]/6} (-a_{02}^2 + \sqrt{4 + a_{02}^2}\,a_{02} - 2) \\ &\quad + 2\Bigg(-\frac{2(4 + a_{02}^2 + \sqrt{4 + a_{02}^2}\,a_{02})x_0 + 4\sqrt{4 + a_{02}^2}\,y_0}{(4 + a_{02}^2 + \sqrt{4 + a_{02}^2}\,a_{02})(-a_{02}^2 + \sqrt{4 + a_{02}^2}\,a_{02} - 2)x_0 + 4\sqrt{4 + a_{02}^2}\,y_0}\Bigg)^{-(-2)^{-1+n}} \\ &\quad \times \Bigg(\Bigg(\frac{a_{02}^2 + \sqrt{4 + a_{02}^2}\,a_{02} + 2}{-a_{02}^2 + \sqrt{4 + a_{02}^2}\,a_{02} - 2}\Bigg)^{[(-2)^n-1]/6} \end{aligned}$
11	$\begin{aligned} x_n &= -\frac{(-x_0 + y_0)^{2^n}}{2} + \frac{(x_0 + y_0)^{2^n}}{2} \\ y_n &= \frac{-[(x_0 - y_0)^2(x_0 + y_0)]^{2^n} + (x_0 + y_0)^{3(2^n)}}{-2(-x_0^2 + y_0^2)^{2^n} + 2[(x_0 + y_0)^2]^{2^n}} \end{aligned}$

Table II. Continued.

No.	Integrated systems
13	$x_n = y_0^{2^n} 2^{[-3(2^n)+3]/2} e^{-(2^n)\pi i}\left(-\sin\left\{\frac{2^n}{4}\left[4\arctan\left(\frac{x_0+2y_0}{2y_0}\right)+\pi-2^{-n}\pi\right]\right\} + \cos\left\{\frac{2^n}{4}\left[4\arctan\left(\frac{x_0+2y_0}{2y_0}\right)+\pi-2^{-n}\pi\right]\right\}\right)\left(\frac{8y_0^2+x_0^2+4x_0y_0}{y_0^2}\right)^{2^{n-1}}$
	$y_n = -\left(\frac{8y_0^2+x_0^2+4x_0y_0}{y_0^2}\right)^{2^{n-1}} y_0^{2^n} 2^{-[3(2^n)+1]/2} \times \cos\left\{\frac{2^n}{4}\left[4\arctan\left(\frac{x_0+2y_0}{2y_0}\right)+\pi-\left(\frac{1}{2}\right)^n\pi\right]\right\} e^{-2^n\pi i}$

and can be easily integrated. In Table II we list also all solutions obtained by this procedure for nontrivial cases.

To summarize, we have found all integrable systems of bidimensional quadratic difference equations with two linear Lie symmetries and we have integrated the nontrivial ones.

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